ELSEVIER Journal of Financial Markets 1 (1998) 253–284

Journal of FINANCIAL MARKETS

Floors, dealer markets and limit order markets

Bruno Biais[a,]*, Thierry Foucault[b], François Salanié[c]

[a] Université de Toulouse, Place Anatole France, 31000 Toulouse, France
[b] Carnegie Mellon University, Pittsburgh, PA 15213, USA
[c] INRA-ERNA, Toulouse, France

Abstract

In dealer markets, liquidity suppliers have entire flexibility to bargain on the price with their customers. In limit order markets, they are restricted to convex schedules: they cannot sell the first share at a higher price than the second. Floor traders simply respond to the liquidity demand conveyed by brokers by crying out one price. In floor markets risk-sharing is inefficient and spreads are large. In dealer markets, risk-sharing can be efficient, but spreads tend to be large. In limit order markets, the unique equilibrium entails efficient risk-sharing and competitive spreads. Hence there is a non-monotonic relation between the efficiency of the market and the extent to which the offers of the liquidity suppliers are restricted.

JEL classification: D43, G1

Keywords: Floor markets; Dealer markets; Limit orders; Market design; Tacit collusion

1. Introduction

Trading market structures differ widely from one exchange to the other. In particular, the way liquidity suppliers post their offers, and the flexibility they enjoy in doing so, vary across markets. On the floors of futures exchanges, brokers announce orders, and traders simply respond by crying out one price. In the NYSE or the Paris Bourse, liquidity suppliers place collections of limit orders. Thus they are restricted to posting convex schedules, i.e., the price at which they sell the first share must be lower than their offer for the second share.

* Corresponding author. E-mail: biais@cict.fr

1386-4181/98/$ – see front matter
PII S1386-4181(98)00003-2

In dealership markets, trades are often the result of a bargaining process between the market makers and the trader demanding liquidity. This bargaining process gives the market makers a large flexibility in the types of offers they can make. For example, they can make take it or leave it offers, or propose quantity discounts. The present paper analyzes how these differences in quoting mechanisms, and the associated restrictions imposed on the offers of liquidity suppliers, lead to differences in terms of prices, bid–ask spreads, trades and risk-sharing efficiency.

We consider a one period symmetric information model whereby (i) one liquidity trader submits an order, (ii) strategic market makers quote offers, and (iii) the liquidity trader selects from these offers, and splits his trade across the market makers, in order to complete his desired trade at the best possible price. In line with the inventory paradigm (see for example Ho and Stoll (1983)), we consider risk averse market makers. This is consistent with the results of many recent empirical studies pointing at the relevance of inventory effects (see for example Madhavan and Smidt (1993), Hasbrouck and Sofianos (1993), Lyons (1995), Manaster and Mann (1996), or Hansch et al. (1997)).

In the dealer market, to reflect the flexibility of the bargaining process, we assume that the market makers can post any price schedule they wish, involving, for example, take it or leave it offers, or they can propose quantity discounts. Reiss and Werner (1995) find empirically that SEAQ dealers do offer quantity discounts. We find that the market makers can use this flexibility to ensure that the equilibrium bid–ask spread is much larger than competitive pricing would entail. Although our analysis is cast in terms of non-cooperative game theory, this can be interpreted as tacit collusion, in line with the empirical findings of Christie and Schultz (1994a,b), and in the spirit of the seminal paper of Wilson (1979).[1]

In the floor market, after the broker has announced the trade size, liquidity suppliers cry out offers that are good up to this amount and thus post linear schedules.[2] The equilibria prevailing in this market structure share a number of features with those arising in the dealer market. In particular, due to tacit collusion, the equilibrium spread can be quite large. Further, the allocation of risk between liquidity suppliers is in general inefficient.

---

[1] Dutta and Madhavan (1997) analyze how long term interaction can enable risk neutral dealers to collude on large spreads. In contrast, we analyze collusion in a one period model with risk averse market makers.

[2] Simultaneously and independently from our work, Simmons and Gerber (1994) have analyzed competition in linear schedules between risk-averse market makers. Our results in Section 4 are similar to theirs, although we offer a more general treatment of the way trades are allocated in the case of ties. Simmons and Gerber (1994) do not analyze the dealer or the limit order market. Neither do they evidence the trade-off between risk-sharing and efficiency which we mention below.

Another similarity between the dealer market and the floor market is that, when comparing across the multiple equilibria which can arise, one is confronted with a trade-off between (i) the efficiency of the allocation of risk between market makers and (ii) the tightness of the spread. Suppose the liquidity trader wants to buy, and consider a candidate equilibrium where risk sharing is very efficient. If one market maker deviated to undercut the others, this deviation would reduce risk sharing efficiency, and thus increase risk bearing for the undercutting dealer. This makes undercutting unattractive and implies that large ask prices (or low bids) will be somewhat immune from undercutting (or over-bidding) if they lead to efficient risk sharing. Consequently more efficient equilibria are more conducive to tacit collusion, which results in larger spreads.

In contrast, in the limit order market, there exists a unique Nash equilibrium, and it involves competitive pricing and efficient risk sharing. The intuition is that, when market makers post limit orders, each market maker can always undercut his competitors until he trades at a price equal to his marginal valuation of the asset. This results in competitive pricing and efficient risk sharing. In contrast with the two other market structures, in the limit order market there is no tacit collusion and no tension between tight spreads and efficient risk sharing.

Overall, our analysis emphasizes that, on theoretical grounds, tacit collusion between market makers is a real possibility, and therefore competitive spreads or efficient risk sharing should not be taken for granted. Further, there is a non-monotonic link between the efficiency of the market structure and the flexibility it leaves to the market makers in designing their offers. When the liquidity suppliers enjoy a very large degree of flexibility, as in the dealer market, they can use it to construct sophisticated quoting strategies supporting tacit collusion on large spreads. Tacit collusion is also likely to emerge in the floor market, where flexibility is the lowest. The limit order market, with its intermediate degree of flexibility, is, in our simple framework, the optimal trading mechanism.

Since we assume the liquidity trader can split her order between market makers (to obtain the best possible price given the schedules that have been posted) our analysis is related to the study of auctions for divisible goods. In a seminal paper, Wilson (1979) shows that in such auctions there exists Nash equilibria where bidders submit demand schedules leading to very low prices, i.e., where tacit collusion emerges. Bernheim and Whinston (1986) extend this intuition to general auctions and strategies. Our result that, in dealer markets or in floors there exists equilibria where market makers coordinate on large spreads is in the spirit of this line of research. Back and Zender (1993) revisit the analysis of Wilson (1979) to analyze an important case of auctions for divisible goods, i.e., Treasury Auctions. One difference between Wilson (1979) and the present paper is our emphasis on discriminatory pricing mechanisms, whereby different market makers could simultaneously trade at different prices, while

Wilson (1979) focuses on uniform price auctions. Another difference is that Wilson (1979) (or Bernheim and Whinston (1986)) do not consider the limit order market. Our modelling framework is also quite different from that of Back and Zender (1993), who study risk-neutral bidders endowed with private signals while we analyze risk-averse market makers without private information.

Our analysis of competition in price schedules is also related to Klemperer and Meyer (1989).[3] The main difference is that Klemperer and Meyer (1989) consider a Walrasian uniform price auction, where supply and demand functions are crossed, while, in our model, the customer selects how much to trade with each of the market makers. This raises the possibility that trades take place simultaneously at different price (which, as shown below, can arise in equilibrium in the dealer market). Maybe the most striking difference between our results and those of Klemperer and Meyer (1989), is that, because of the different structure of the game, in the limit order market we find that equilibrium is unique.[4]

The next section describes our model and assumptions. In the third section the economic intuition of our main results is presented with a numerical example. Section 4 analyzes price formation, risk and trades in the floor market. The limit order market is analyzed in Section 5, while the case of the dealer market is studied in Section 6. Section 7 offers concluding comments. Proofs not in the text are in the appendix.

2. The model

2.1. Agents and preferences

Consider the market for a risky asset, with end of period liquidation value $v = \mu + \varepsilon$, where μ is a constant and $\mathrm{E}(\varepsilon) = 0$.

There is one liquidity trader. She can be a buyer or a seller. For brevity, we hereafter focus on the case of a buyer. The case of a seller is symmetric. The utility function of the liquidity trader is lexicographic. She *must* buy D, but under this constraint seeks to minimize the price paid for these shares. If the liquidity trader fails to buy D, her utility is $-\infty$. If she purchases D (or more) against total payment t, her utility is $K - t$, where K is a constant.

N suppliers of liquidity (also referred to as market makers) compete for the order flow from the liquidity trader.[5] They are strategic, rational and risk averse.

[3] See also Grossman (1981).

[4] Equilibrium uniqueness in a limit order market game obtains also in Biais et al. (1997), for the case of risk neutral market makers facing privately informed investors.

[5] Abusing notations a little, N will also denote the set of market makers.

Market maker i has endowment I_i in the risky asset and C_i in cash. There is no asymmetric information about the value of the asset, the positions of the market makers or the liquidity demand. The expected utility of market maker i if he does not trade is: $\mathrm{E}\mathscr{U}_i(I_i v + C_i)$ which we will denote $\mathscr{U}_{i,0}$. The expected utility of market maker i if he sells q_i in exchange for cash transfer t_i is: $\mathrm{E}\mathscr{U}_i((I_i - q_i)v + C_i + t_i)$. Let $V_i(q_i)$ be the reservation transfer of market maker i for quantity q_i. It is the transfer such that he is indifferent between selling q_i or not trading:

$$\mathscr{U}_{i,0} = \mathrm{E}\mathscr{U}_i((I_i - q_i)v + C_i + V_i(q_i)).$$

It is straightforward to show that concavity of the utility function implies that V_i is convex with respect to q. Totally differentiating the above equation, we have that

$$0 = \mathrm{E}[(V_i'(q_i) - v)\mathscr{U}_i'((I_i - q_i)v + C_i + V_i(q_i))].$$

Differentiating totally again

$$V_i''(q_i) = -\frac{(V_i'(q_i) - v)^2\mathscr{U}_i''((I_i - q_i)v + C_i + V_i(q_i))}{\mathrm{E}[\mathscr{U}_i'((I_i - q_i)v + C_i + V_i(q_i))]} > 0.$$

Convexity of V means that the marginal valuation of market maker i for the asset increases as the amount he sells increases. This is because large sales lead to unbalanced positions making the market maker reluctant to sell more. As will be shown below, the property that $V_i''(q_i) > 0$ plays an important role in our analysis.

For simplicity and without consequence for our qualitative results, we focus on the case where ε is normal with standard deviation σ, and the market makers have CARA utilities, $\forall i = 1, 2, \dots, N$, $\exists \gamma_i > 0$, $\mathscr{U}_i(x) = -\exp(-\gamma_i x)$. The reservation expected utility of market maker i is

$$\mathscr{U}_{i,0} = \mathscr{U}_i\left(C_i + I_i\mu - \frac{\gamma_i\sigma^2}{2}I_i^2\right)$$

while his expected utility when he trades is

$$\mathscr{U}_i(C_i + t_i + (I_i - q_i)\mu - \frac{\gamma_i\sigma^2}{2}(I_i - q_i)^2) = \mathscr{U}_{i,0}\exp^{-\gamma_i[t_i - V_i(q_i)]}$$

where

$$V_i(q_i) = q_i\left[\mu - \frac{\gamma_i\sigma^2}{2}(2I_i - q_i)\right]. \tag{1}$$

The difference between the payment received by market maker i (t_i), and his valuation for the asset (V_i) is a measure of his gains from trade. To emphasize this, we denote the utility of market maker i if he trades q_i as

$$U_i(t_i - V_i(q_i)) = \mathscr{U}_{i,0}\exp^{-\gamma_i[t_i - V_i(q_i)]}$$

2.2. Social welfare and the first best allocation

Before we turn to our positive analysis of existing market structures, it is useful to characterize the first best allocation in our simple model. It provides a benchmark against which to compare the outcomes reached in the different market structures we analyze. To do so, we consider a utilitarian measure of social welfare, equal to the sum of the utility of the liquidity trader and the gains from trade of the market makers.[6] If the liquidity trader cannot exchange D, social welfare equals $-\infty$. If the liquidity trader is able to buy D, then social welfare is equal to

$$\left[\sum_{i=1}^{N} t_i - V_i(q_i)\right] + \left[K - \sum_{i=1}^{N} t_i\right].$$

Hence, the first best is the solution of

$$\max_{\{t_i, q_i\}_{i=1,\dots,N}} \left[\sum_{i=1}^{N} t_i - V_i(q_i)\right] + \left[K - \sum_{i=1}^{N} t_i\right], \text{ s.t. } \sum_{i=1}^{N} q_i = D.$$

Since the gains from trade are linear in monetary transfers, the first best allocation depends only on quantities and not on transfers. Note that large transfers are equivalent to large spreads. Hence, focusing on the efficiency of risk sharing downplays the importance of the bid–ask spread. In the following analysis we will also analyze the spread in relation to the structure of the market. In fact we will show (in Proposition 11) that a conflict can arise between the tightness of the spread and the efficiency of the allocations.

The first best allocation of risks is such that all the marginal valuations are equal across market makers. In fact, the vector of optimal trades characterizing the first best is the competitive outcome. The latter is characterized by the price p^* and allocation $\{q_1^*, \dots, q_N^*\}$ such that $\sum_i q_i^* = D$ and $\forall i$, $V_i'(q_i^*) = p^*$. Because valuation functions V_i are convex, this allocation exists and is unique. Hereafter, for simplicity, we will assume that the competitive allocation entails that at least 2 market makers participate to the trade.

2.3. Market structures

The first market structure we consider is a *floor market*, where brokers announce the quantity their customer desires to trade, and floor traders respond by announcing the prices at which they are willing to accomodate this quantity. Then the broker allocates the order to the trader (or traders) posting the best price. This is as in the open-outcry trading process used in futures markets. Since

[6] Similar insights could be obtained by reasoning in terms of Pareto optimality instead.

traders simply respond with a price to the announced quantity, they implicitly commit to accept, at this price, any trade up to the announced quantity. This implies that the schedules offered by the market makers are *linear* in the quantity they trade:[7]

$$t_i(q_i) = p_i q_i.$$

In other market structures, liquidity suppliers are able to post more complex schedules. In the Paris Bourse, the New York Stock Exchange or the Tokyo Stock Exchange liquidity suppliers post limit orders. A sell (buy) limit order specifies an ask price (bid price) and the maximum quantity the liquidity supplier is willing to sell (buy) at this price. The second market structure we consider is a *limit order market*. In a limit order market, the supply of liquidity is designed by the market makers in response to market conditions, and in particular in response to the expressions of liquidity demand. The empirical analysis of the dynamics of the limit order book in the Paris Bourse by Biais et al. (1995) illustrates this. For example they find that, to express their demand for liquidity, traders can place an 'ordre au prix du marché' or a limit order between the best quotes, which frequently attracts liquidity supply, under the form of competing limit orders from the other side of the market. Thus, in order to model the limit order market, we assume the following:

- First, the liquidity trader announces his desired trade (D).
- Then, each liquidity supplier places a collection of limit orders. This set of offers posted by each market maker defines a schedule $t_i(.)$ which gives, for each order of size q_i, the transfer $t_i(q_i)$ required by market maker i in order to execute the order. It's worth stressing that $t_i(.)$ must be *convex* in the limit order market. This reflects price priority and the fact that $t_i(.)$ is the transfer schedule associated to a collection of limit orders. Note that, by definition, the different offers of one liquidity supplier are exclusive: the liquidity trader cannot select simultaneously two different offers from the same market maker.
- Finally, given these offers, the liquidity trader allocates his order in order to minimize his total payment, i.e. $\{q_i\}_{\{i=1,\dots,N\}}$ solves

$$\begin{aligned} &\mathrm{Min}_{(q_i)} \sum_i t_i(q_i) \\ &\sum_i q_i \geqslant D. \end{aligned} \tag{2}$$

[7] Ho and Stoll (1983), Subrahmanyam (1991) and Biais (1993) analyze competition in linear prices between risk averse market makers, under the assumption that the liquidity trader has to execute the *entire* trade with *only one* market maker. In this case the best ask price is the reservation price of the second best market maker. We will show below that significantly different results obtain when the trader can split his orders among the market makers.

The third market structure we consider is a *dealership market*. In dealer markets, such as SEAQ, NASDAQ or the foreign exchange market, trades frequently occur at prices differing from the quotes previously posted on screens (see, e.g., (Reiss and Werner (1995)). Rather transaction prices are negotiated on the phone. This bargaining process goes as follows. The customer calls the dealers and announces his liquidity need. Then each dealer announces on the phone the conditions at which he is willing to trade. Finally the customer decides if and how much to trade with each dealer. The exact workings of this bargaining process are left to the market makers to decide. This leaves them a large flexibility in the type of offers they can make. In particular, they can offer quantity discounts, and sell larger amounts at a lower unit price than lower amounts (Reiss and Werner (1995)). These quantity discounts can be so pronounced that in practice small trades may be deterred.

To model the dealership market, and capture as simply as possible the richness of this bargaining process, we assume that the liquidity trader first announces his desired trade (D). Second each dealer i proposes a menu of offers $t_i(.)$. Finally, as in the limit order market, the liquidity trader allocates his order among the different dealers, with a view at minimizing the total cost of purchasing D. There are two differences between the limit order schedules and the menus of trades offered in the dealer market.

- First the transfer schedules must be convex in the limit order market but *not necessarily* in the dealership market. This entails that market makers can offer quantity discounts in the dealership market while they cannot in the limit order market.
- Second, in the limit order market, if market maker i offers 4 shares at price 14, then the liquidity trader can decide to buy any number of shares from 0 to 4 at this price. In the dealer market, in contrast, we assume the market makers have entire flexibility in designing their offers. Consequently, they can specify the exact number of shares they are willing to trade at each price. Market maker i, for example, can offer to sell 4 shares at price 14, while refusing to sell 1 or 2 shares at this price. This is in line with the casual observation that, on SEAQ international, small traders can find it very hard to obtain attractive quotes for small amounts, while dealers readily offer liquidity for the larger Normal Market Size.

These differences reflect the greater flexibility offered by the bargaining process to the dealers in the dealership market.

Note that, both in the dealer market and the limit order market, trading can take place simultaneously at different prices. In the dealer market this arises when the trader splits his order between dealers offering different ask prices. In the limit order market this corresponds to the case where the order walks up the book, and hits orders placed at different prices. This gives the market structures we consider a discriminatory pricing flavor. This differs from the uniform pricing call auction used to open the market in the Paris Bourse and the NYSE.

To complete the description of the game in the three market structures, we need to specify what happens if the offers of the market makers are such that it is not possible to buy D. In this case we assume that no trade takes place. Such an outcome will never happen in equilibrium, however, since one market maker would be better off offering D at a sufficiently high price. We also need to specify what happens if there are several ways for the liquidity trader to allocate his order among the different market-makers, while keeping the total payment unchanged. In the case of such ties, we assume the liquidity trader follows a given allocation rule, specified by the rules of the market. Note that ex-post the trader is indifferent between any allocation. We will study how variations in the allocation rule can significantly affect the equilibria. In each market structure, we study the Nash equilibria, where market makers simultaneously post menus of offers, which are best responses to the offers of their competitors. For simplicity, we restrict our attention to pure-strategy equilibria. Hence the strategy space of a market maker is identified to the set of offers he is allowed to make.

Finally, note that in the limit order market, liquidity suppliers can post the same offers as in the floor market while the converse is not true. In the same way, in the dealership market, liquidity suppliers can post limit orders but are able to use a richer set of offers. In this sense, the dealership market is the market structure, which offers the largest flexibility to the dealers with respect to the design of their offers whereas the floor market offers the smallest flexibility. We will consider in turn the different market structures, proceeding from the least flexible to the most flexible.

3. A numerical example

To offer an intuitive survey of our results we present in this section a numerical example illustrating the main features of the equilibria derived in the general case in the following sections. Assume the liquidity trader wants to purchase 4 shares. Liquidity is supplied by two market makers. The reservation values placed by market maker 1 on the first, second, third and fourth units are 10, 11, 12 and 23, respectively, while the corresponding reservation values for his competitor are 12, 13, 14 and 25. These valuations are represented in Fig. 1. Note that market maker 1 has a lower valuation for the asset than his competitor, reflecting for example a larger position in the asset. In this example, the optimal allocation of risk is such that market maker 1 sells three shares while market maker 2 sells one share. This corresponds to the competitive equilibrium, and the resulting marginal valuations of the market makers are equal to 12, the competitive equilibrium price.

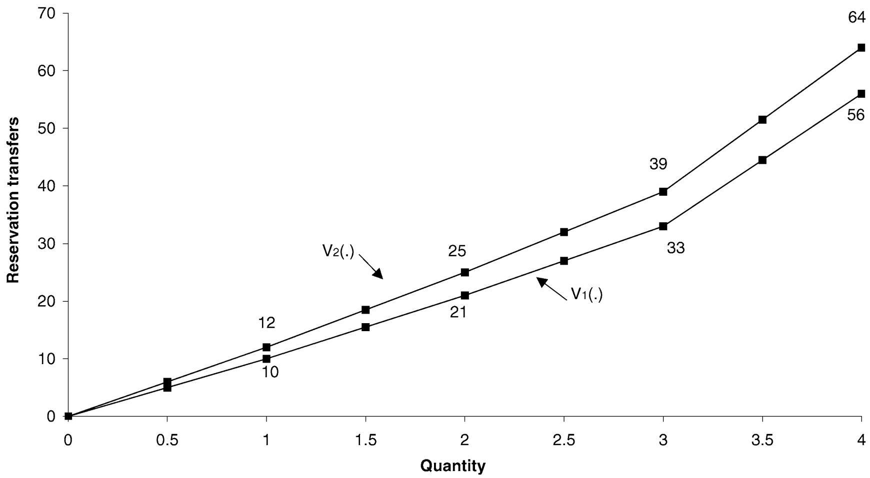

Fig. 1. A simple example with two market makers.

First consider the floor market where market makers post linear schedules, and suppose that in case of ties the order is split equally between the two market makers. A Nash equilibrium in this market is when both market makers quote the same ask price 14 and sell two shares each. This equilibrium ask price is significantly above the competitive price and the allocation of risks between the market makers is not optimal. Yet, none of the two market maker finds it attractive to undercut his competitor. Consider for example market maker 1, and suppose he quotes an ask price slightly below 14. In this case he would attract all the order flow and would sell 4 shares, for a total payment of 56. This would leave him with 0 surplus however, while in the Nash equilibrium his surplus equals 7. Selling 4 shares is unattractive for the market maker, because such a large trade leads to a large risk exposure, reducing his total gain from trade.

Second, consider the limit order market, and assume that in case of ties market makers receive a share of the total order flow prorata to the size of their order. The unique equilibrium in this market is when both market makers offer to sell at the competitive equilibrium price (12). At this price, market maker 1 offers to sell up to 12 shares, while market maker 2 offers to sell 4 shares. Because of prorata allocation, market maker 1 ends up selling 3 shares and his competitor one share, i.e., their competitive supply. To see that market maker 1 has no incentive to deviate from this equilibrium strategy, first note that at the competitive equilibrium price his preferred trade is his competitive equilibrium trade. Furthermore, if he posts a price higher than 12, his competitor serves the entire order, and he forfeits all gains from trade. Finally if market maker 1 undercut his competitor by quoting a price below 12 he would obtain price priority. In this case, his preferred trade would be to sell less than 3 shares, however. This would leave him with a lower surplus than at the Nash equilibrium. The same argument shows that market maker 2 has no incentive to deviate as well. Finally consider the dealer market. Assume that in case of ties the allocation rule favors splitting orders between market makers rather than allocating the entire trade to a single market maker. In this case there is a Nash equilibrium where market maker 1 offers to sell 1 share at price 12 or 4 shares at 14.25 each, while market maker 2 offers to sell 3 shares at price 15 or 4 shares at 14.25. This illustrates how, as discussed in the previous section, in the dealership market, market makers can offer quantity discounts. In this case the trader buys one share from market maker 1 at 12 and three shares from market maker 2 at price 15. Note that, although market maker 1 is more efficient than his competitor, in this equilibrium he only sells one share. Could he undercut market maker 2 and attract a larger fraction of the order? To do so he would have to sell 4 shares. He cannot sell these at a price higher than 14.25 each, corresponding to a total transfer of 57. Now, his valuation for these 4 shares is equal to 56. Hence, if he were to undercut his competitor market maker 1 would only obtain a surplus equal to 1, which is lower than his equilibrium surplus. Finally note that market maker 1 cannot offer to sell the first share at more than 12, else the customer would buy the four shares from market maker 2.

This numerical example illustrates a number of features which we will show below to be general:

- The equilibria arising in the different market structures are quite different.
- In the limit order market the competitive equilibrium is the unique equilibrium. This contrasts with the equilibrium multiplicity arising in the two other market structures.
- The risk sharing efficiency and competitive pricing prevailing in the limit order market contrast with the inefficiencies arising in the two other market structures. This suggests that it is not desirable to leave too much flexibility to the liquidity suppliers (as in the dealer market) or too little flexibility (as in the floor market).
- Because of the discriminatory pricing nature of the market structure we consider, in the dealer market, equilibrium trades can be conducted simultaneously at different prices.
- *Bertrand pricing*, whereby the market makers would trade at their reservation prices does not arise in general.

4. The floor market

In this section, we consider the floor market whereby, the liquidity trader's broker announces his desired trade D, and then each market maker i announces a price p_i at which he stands ready to serve demand up to D. Were the demand indivisible, this would be the framework of the analysis of Ho and Stoll (1983) or Subrahmanyam (1991). In this case, prices are set as in a second price auction, and equal to the reservation price of the second best seller (Bertrand pricing). We extend this analysis to the case where the demand can be split between the market makers. In addition to analyzing the consequences of trade splitting, we also analyze the impact of the allocation (or priority) rule used in case of ties, i.e., when there is more than one way for the buyer to purchase D at the minimum cost.

A simple allocation or priority rule is *equal-splitting*, whereby the demand is equally shared between the best quoters.[8] Note that this rule is deterministic. Stochastic rules may also be considered. The simplest stochastic allocation rule is the *non-splitting* rule, which allocates the entire demand with equal probability to any of the market makers quoting the best price. This is similar to the indivisible good case. In addition to the non splitting and the equal splitting rules, there exists a variety of stochastic rules, in which the quantity sold by each of the m best quoters is a random variable, q_m. In the following, E will denote the expectation operator, taken over this variable.

Consider a candidate equilibrium, where $m \geqslant 2$ market makers in a subset J of N quote the best price p, while the other market makers quote strictly greater prices. The expected utility of market maker j in the set J of best-quoters is $\mathrm{E}U_j(pq_m - V_j(q_m))$. For the market maker to be willing to quote p rather than a larger price at which he would not trade, we need

$$\mathrm{E}U_j(pq_m - V_j(q_m)) \geqslant \mathscr{U}_{j,0} \tag{3}$$

or

$$\mathrm{E}e^{-\gamma_j(pq_m - V_j(q_m))} \leqslant 1. \tag{4}$$

The left-hand side of Eq. (4) is continuous and decreasing in p. For p low enough, it is larger than 1, and as p goes to infinity it goes to 0. Consequently, we can state the following lemma.

Lemma 1. *There exists a price* $\underline{p}_j^m$ *which solves*

$$\mathrm{E}e^{-\gamma(\underline{p}^m{}_j q_m - V_j(q_m))} = 1$$

and such that market maker j is willing to share the trade with $m-1$ *other market makers rather than not participating to the trade iff* $p \geqslant \underline{p}_j^m$.

$\underline{p}_j^m$ can be interpreted as the reservation valuation of market maker j, since at this price j is indifferent between sharing the trade with $m-1$ other market makers

[8] It is the allocation rule considered in Simmons and Gerber (1994) and Dastidar (1994).

or not trading. This reservation price varies as the number of market makers sharing the trade varies, and as the allocation rule to be used varies, since they both influence the trade of market maker j.

On the other hand, market maker j could also quote a price p' slightly below p and serve all the demand at this price. Such undercutting would generate utility: $U_j(p'D - V_j(D))$. Taking the limit as p' goes to p, the condition under which j prefers not to undercut is

$$\mathscr{U}_{j,0}\mathrm{E}\mathrm{e}^{-\gamma_j(pq_m - V_j(q_m))} \geqslant \mathscr{U}_{j,0}\mathrm{e}^{-\gamma_j(pD - V_j(D))} \tag{5}$$

or

$$\mathrm{E}\mathrm{e}^{-\gamma_j(p(q_m - D) - V_j(q_m) + V_j(D))} \leqslant 1. \tag{6}$$

The left-hand side of Eq. (6) is continuous and increasing in p. For p low enough it is lower than 1, and as p goes to infinity it also goes to infinity. Consequently, we can state the following lemma.

Lemma 2. *There exists a price* $\bar{p}_j^m$ *which solves*

$$\mathrm{E}\mathrm{e}^{-\gamma(\bar{p}^m{}_j(q_m - D) - V_j(q_m) + V_j(D))} = 1$$

and such that market maker j prefers to share the trade with $m - 1$ *other market makers rather than undercutting iff* $p \leqslant \bar{p}_j^m$.

Finally, note that $\underline{p}_j^m < \bar{p}_j^m$ if Eq. (6) holds for $\underline{p}_j^m$, i.e., $1 < \mathrm{e}^{-\gamma(\underline{p}^m{}_j D - V_j(D))}$, that is,

$$\underline{p}_j^m < V_j(D)/D,$$

which holds, since the convexity of the valuation function implies that $\underline{p}_j^m$, the price at which j is indifferent between sharing the trade or not trading, is lower than $V_j(D)/D$, the price at which the market maker is indifferent between serving all the demand or not trading. The following proposition is a direct implication of the above discussion.

Proposition 3. *There exists an equilibrium where the m market makers in J* (*a subset of N, with* $m \geqslant 2$) *quote price p while the other market makers quote strictly larger prices iff*

$$\forall j \in J, \quad \underline{p}_j^m \leqslant p \leqslant \bar{p}_j^m \tag{7}$$

and

$$\forall j \notin J, \quad p \leqslant \underline{p}_j^{m+1}.$$

Except when the price is equal to the lower bound of the interval $[\underline{p}_j^m, \bar{p}_j^m]$, equilibrium prices are above reservation prices, in contrast with the indivisible demand case. To see the intuition for this result consider the case of two market makers. Suppose one of them quotes a rather large price. If the other matches this quote both market makers earn large profits. Should the other undercut this quote? If he does so then he serves the whole demand D. Now, this can be very unattractive since marginal valuations are convex, or, to put it differently, since this would entail large risk bearing. Consequently it may well be that the market maker is better off matching the quote of his competitor, rather than undercutting it, even if this quote is rather large. Thus when liquidity suppliers are risk averse and when they can share trades, Bertrand pricing cannot be taken as granted. As shown in the next sections, this is a property common to all market structures. The size of the markup between dealers' reservation prices and dealers' quotes depends on the market structure, however. Note that since $\underline{p}_j^m < \bar{p}_j^m$ there is a multiplicity of equilibria, which creates a coordination issue: the two dealers must coordinate on one of the many quotes at which they earn profits and at which they are better off matching than undercutting.

Further, there is a trade off between optimal risk sharing and the tightness of the spread in the floor market. This can be seen by comparing the results obtained with the equal splitting rule and the non splitting allocation rule. For simplicity, we assume that the market makers are identical and we focus on equilibria in which they all post the best price.

In the case where the allocation rule entails equal splitting, the trade of market maker j if he shares the order with $N-1$ other market makers is no longer stochastic. In this case the cut-off prices simplify to

$$\underline{p} = V(D/N)/(D/N)$$

and

$$\bar{p} = \frac{V(D) - V(D/N)}{D - D/N}$$

where we have dropped the subscript j and the superscript m since all the market makers are identical and participate to the trade. Hence for p to be an equilibrium, it must be such that

$$\underline{p} = \frac{V(D/N)}{D/N} \leqslant p \leqslant \bar{p} = \frac{V(D) - V(D/N)}{D - D/N}. \tag{8}$$

In the case of the non-splitting allocation rule (which allocates the entire trade to one of the best quoters), the cut-off prices simplify to

$$\underline{p} = V(D)/D = \bar{p} \tag{9}$$

and, as discussed above, the equilibrium price is the same as in a second price auction. Clearly, in terms of risk sharing, the non splitting allocation rule is dominated by others, in particular the equal splitting rule, which shares the trade, and thus spreads risk exposure across market makers. Does this lower risk sharing efficiency translate in less attractive prices for the buyer? Comparing the expressions in Eqs. (8) and (9), the equilibrium price under the non splitting rule is lower than the largest possible equilibrium price under the equal splitting rule. Consequently although the equal splitting rule is more efficient with respect to risk sharing than the non splitting rule, and thus imposes lower costs on the market makers, it does not necessarily lead to better prices for the buyer. The intuition is as follows. The equal splitting rule leads to better risk sharing than the non splitting rule. Because they share risk better in equilibrium, the market makers are less eager to deviate from this equilibrium and undercut each other, since it would lead to a less efficient allocation of trades. This softens the competitive pressures and consequently this leads to larger prices. This result is reminiscent of Wilson (1979).

The last issue we investigate is whether the high equilibrium prices obtained in this section would go to competitive prices, if the number of competing market makers went to infinity. To see that it does not, note that the limit of the right-hand side of Eq. (8) as N goes to infinity is

$$(V(D) - V(0))/D.$$

This can be rather large and is definitely above the competitive price. The reason why equilibrium prices may not go to the competitive price as the number of market makers goes to infinity is the following. Suppose a very large number of market makers is present and all quote a very large price. In equilibrium, because the number of market makers is large, each market maker executes a small fraction of the trade. In contrast, if one individual dealer were to undercut his competitors, he would have to serve the whole demand D and this would lead to a large increase in risk bearing for this market maker. Hence the same reasoning as above applies, unaltered by the large number of competing market makers.

5. The limit order market

In the floor market, liquidity suppliers can only quote one price. We now turn to the limit order market, where each market maker can post a collection of limit orders, i.e., price and quantity pairs. We analyze the consequences for equilibrium outcomes of this richer strategy space available to the market makers.

Market maker i will be referred to as *indispensable* if, in equilibrium, the buyer is worse off not trading with him. Furthermore, market maker i is said to be *active* if his offers are such that he is eligible to take part to the trade. Also, denote $\mathscr{A}$, the set of active market makers.

The following proposition characterizes the equilibrium arising in the limit order market:

Proposition 4. *Necessary conditions for the offers* $\{p_i, \bar{q}_i\}_{i=1,\dots,N}$ *to form an equilibrium in the limit order market are that*

- *all active market makers quote the competitive price,* p^*,
- *no active market maker is indispensable,*
- *and for these offers the allocation rule selects the competitive allocation* $\{q_i^*\}_{i\in N}$.

Note that the competitive price p^* is strictly above dealers' reservation prices. Thus as in the floor market, Bertrand pricing is not obtained in the limit order market. However, strikingly, while in the floor market there are many inefficient or high prices equilibria, in the limit order market there is a unique equilibrium outcome and it involves competitive pricing and efficient risk sharing. The intuition is the following. As explained in the previous section, in the floor market, the market makers are unwilling to undercut each other since it would lead them to serve the whole demand, which is costly in term of risk bearing. In the case of limit orders, in contrast, the market makers can undercut the quotes of their competitors while protecting themselves from this large risk bearing, by setting a cap to the quantity they accept to sell. Furthermore, for any given price, as long as one market maker does not sell his competitive supply, he has incentives to undercut his competitors to attain his competitive supply. Consequently, the limit order environment is very conducive of undercutting, and drives prices down to their competitive level.

In the equilibrium described in Proposition 4, the allocation rule to be used in the case of ties is rather important, since the non-indispensability condition implies that in equilibrium there must be a tie and moreover the allocation rule must select the competitive allocation. As stated in the next corollary, under the prorata allocation rule, equilibrium does indeed exists.

Corollary 5. *If the prorata allocation rule is used in the case of ties, then equilibrium exists in the limit order market.*

There are many other allocation rules than prorata under which equilibrium in the limit order market exists. The only requirement imposed by Proposition 4 is that for each competitive allocation $\{q_i^*\}_{i\in N}$ one can find a set of limit quantity $\{\bar{q}_i\}_{i\in N}$ such that the allocation rule maps $\{\bar{q}_i\}_{i\in N}$ into $\{q_i^*\}_{i\in N}$. Although this requirement leaves a lot of flexibility in the choice of the allocation rule, one cannot use *any* rule. There are some rather natural rules for which equilibrium fails to exist. Two examples are the following:

- *Quantity priority*: For simplicity consider the case of two market makers quoting the competitive price. Under quantity priority the market maker (say 1) who offers the largest limit quantity ($\bar{q}_1 > \bar{q}_2$) is fully executed ($q_1 = \bar{q}_1$), while the other market maker gets the residual demand ($D - q_1$).

Under this rule, the last condition of Proposition 4 implies that market-maker 1 must offer a limit quantity equal to his competitive trade, so that if the latter is lower than the demand (D), market maker 2 is indispensable, which is not consistent with the second necessary condition for equilibrium (indispensability). Hence under this rule equilibrium does not exist.

- *Equal splitting*: Under this rule (when it is possible) the buyer splits demand equally between market makers quoting the best price. This sounds like a fairly natural rule, and is the standard rule considered in linear pricing competition models. Under this rule, equilibrium does not exist in the limit order market, however. This is because (i) equal splitting generically does not allow one to generate competitive allocations and (ii) if market makers were to post limit quantities equal to their competitive supply, to avoid equal split, then each would be indispensable, which is impossible in equilibrium.

For many allocation rules for which equilibrium exists (including prorata) there is an infinity of sets of limit quantities $\{\bar{q}_j\}_{j\in N}$ which yield the competitive allocation $\{q_j\}_{j\in N}$. Hence market makers must coordinate on one. In practice, coordination failures can arise in such cases.

Note that, although in the limit order market each liquidity supplier can place a collection of orders, in equilibrium each market maker needs to post only one offer (on the buy side, and another on the sell side of the market). Also note that Proposition 4 is not unlike the result obtained by Dubey (1982) in the somewhat different context of general equilibrium implementation. Dubey (1982) considers uniform pricing with a Walrasian auctioneer, however. In contrast we allow in the limit order market for simultaneous trades at different prices. Yet we show that in equilibrium all trades are conducted at the competitive price. Finally note that one can interpret the results presented in Proposition 4 and Corollary 5 in terms of optimal mechanisms. Consider a benevolent central planner, who would not know the types of the agents (summarized by D, and $\{I_i\}_{i=1,\dots,N}$) but only their distribution, and who would endeavor to implement the first best. As mentioned in Section 2 this is the competitive allocation. Now, Proposition 4 and Corollary 5 state that the unique equilibrium outcome in the limit order market is the competitive outcome. Hence the central planner can achieve unique Nash implementation of the first best by structuring the trading mechanism precisely as the limit order market we analyze, with prorata allocation in case of ties.

Turning from the floor market to the limit order market involved an increase in the richness of the strategy space available to the market makers. This led to an increase in the risk sharing efficiency of the market. In the next section we turn to the dealer market, whereby the market makers can use even more complex strategies. We investigate the consequences of this further increase in complexity on the efficiency of the market.

6. The dealer market

6.1. Equilibrium conditions

Given her competitors' offers, market maker i selects the schedule maximizing her expected utility. In fact, the schedules of her competitors and the liquidity demand D define for market maker i a residual inverse demand function. For any quantity q_i that market maker i would like to trade, there exists a maximum transfer that she can ask for, while making sure that she will indeed sell q_i.[9]

In order to establish this more explicitly, let us denote $A_{-i}(q)$ the minimum payment for which the liquidity trader can buy quantity q from all N market makers except i. Now suppose i would like to sell q_i. To make sure she will conduct this trade, i can make a take it or leave it offer to sell q_i against payment t_i. Two cases can occur. Either the liquidity trader can buy the quantity $D - q_i$ from the other market makers or he cannot. In the second case, were the buyer to accept i's offer, then the buyer would have to buy more than D. But then i would be better off lowering q_i, while *asking for the same payment*, so that the buyer would only need to buy D. Thus we only need to consider take it or leave it offers (t_i, q_i) such that exactly $D - q_i$ can be purchased from the other market makers. The buyer will accept such a take or leave it offer, as long as

$$A_{-i}(D - q_i) + t_i \leqslant A_{-i}(D). \tag{10}$$

To obtain the best possible price i will saturate inequality (10) and set[10]

$$t_i = A_{-i}(D) - A_{-i}(D - q_i). \tag{11}$$

Given the 'inverse demand curve' defined in Eq. (11) market maker i can select an optimal trade by placing a single take or leave it offer, for quantity q_i solving

$$\mathrm{Max}_{q_i}\, U_i(A_{-i}(D) - A_{-i}(D - q_i) - V_i(q_i)). \tag{12}$$

Hence we can state our first result (which is similar to Bernheim and Whinston (1986)):

Lemma 6. The equilibrium trades of dealer i must solve

$$\mathrm{Min}_{q_i}\, A_{-i}(D - q_i) + V_i(q_i).$$

---

[9] This approach in terms of residual demand is similar to Wilson (1979) and Back and Zender (1993).

[10] Strictly speaking, making the constraint bind leads to a tie, since the buyer could also exclude market maker i and bear the same total payment. To avoid such complications, market maker i may ask $A_{-i}(D) - A_{-i}(D - q_i) - \varepsilon$ ($\varepsilon > 0$ and arbitrarily small), so that the allocation rule does not matter anymore.

Lemma 6 provides a characterization of the equilibrium *quantities* traded. We now discuss further the equilibrium *transfers*. Consider a candidate equilibrium characterized by schedules $\{t_i(.)\}_{i=1,\dots,N}$ and by actual trades and transfers $\{t_i, q_i\}_{i=1,\dots,N}$ such that $\sum_{i=1,\dots,N} q_i = D$. Suppose that in this candidate equilibrium, the total price paid by the buyer $F = \sum_{i=1,\dots,N} t_i$ is strictly lower than the price he would pay if he did not trade with i: $A_{-i}(D)$. In this case, market maker i is *indispensable*. But then, market maker i would be better off deviating from his candidate equilibrium offer, and post a single take or leave it offer, for quantity q_i against transfer $t_i + \varepsilon$, where ε is a small positive number, lower than $A_{-i}(D) - F$. Hence it cannot be an equilibrium that $A_{-i}(D) > F$. On the other hand, if $F > A_{-i}(D)$, then F cannot be the equilibrium payment, since the buyer would be better off buying at $A_{-i}(D)$. Consequently, as stated in the next lemma, in equilibrium $A_{-i}(D) = F$.[11]

Lemma 7. *In equilibrium, no dealer can be indispensable, in the sense that*

$$A_{-i}(D) = F \quad \forall i \tag{13}$$

where F is the total price paid by the buyer in equilibrium.

Building on Lemmas 6 and 7, we can state the following proposition:[12]

Proposition 8. *Schedules* $\{t_1(\cdot), \dots, t_n(\cdot)\}$ *form an equilibrium if and only if they generate outcomes such that no dealer is indispensable and, the trade of each dealer i* solves

$$\mathrm{Min}_q\, A_{-i}(D - q) + V_i(q).$$

An important ingredient in the reasoning leading to the equilibrium conditions stated in Proposition 8, is that each market maker can post a single take-or-leave it offer (solution of Eqs. (11) and (12)) as a best response to the quotes of her competitors. This enables one to pin down the equilibrium *outcomes*. Note however that to describe the equilibrium *strategies* supporting the set of equilibrium outcomes, one must consider at least another quote placed by each market maker. If each market maker posted only one take-it-or-leave-it offer, corresponding to his equilibrium trade, this would raise the possibility that some market makers would be indispensable, in contradiction with Lemma 7. This point is discussed further in Section 6.3 below, where the equilibrium strategies are characterized.

[11] As Lemma 6, this result is in the same spirit as Bernheim and Whinston (1986).

[12] That Conditions (a) and (b) in Proposition 8 are necessary stems from Lemmas 6 and 7. The sufficiency of (a) and (b) is straightforward and omitted for brevity.

6.2. An explicit necessary condition for equilibrium

In order to better characterize the possible equilibria we now provide explicit necessary conditions on equilibrium outcomes, that is conditions which do not involve the implicit residual inverse demand function $A_{-i}(.)$. Consider an outcome $(t_i, q_i)_{i\in N}$. In equilibrium, market maker i obtains expected utility

$$U_i(t_i - V_i(q_i)). \tag{14}$$

In equilibrium market maker i is better off selling q_i than not trading, i.e.,

$$\forall i \in N, \quad t_i \geqslant V_i(q_i). \tag{15}$$

Now, consider a set J of active market makers to which i belongs: $i \in J \subset N$. If i wanted to undercut this set of market makers, and sell instead of them, he would have to offer to sell $\sum_J q_j$ for a transfer slightly lower than $\sum_J t_j$. In this case i would obtain expected utility

$$U_i\left(\sum_J t_j - V_i\left(\sum_J q_j\right)\right). \tag{16}$$

In equilibrium, no market maker has incentive to undercut her competitors, hence the utility in Eq. (14) must be larger than the utility in Eq. (16). This implies that

$$\forall i \in J \subset N, \quad \sum_{J-\{i\}} t_j \leqslant V_i\left(\sum_J q_j\right) - V_i(q_i). \tag{17}$$

The LHS of Eq. (17) is the increase in revenues due to undercutting, the RHS is the increase in risk bearing. The market maker i does not undercut if the latter is larger than the former. This leads to the following proposition:

Proposition 9. *Equilibrium outcomes* $\{(t_i, q_i)\}_{i=1,\dots,N}$ *must satisfy*

$$0 \leqslant \sum_{j\in J, j\neq i} [t_j - V_j(q_j)] \leqslant V_i\left(\sum_{j\in J} q_j\right) - \sum_{j\in J} V_j(q_j) \qquad \forall J \subset N \;\; \textit{and} \;\; \forall i \in J. \tag{18}$$

Condition (18) implies that

$$\sum_J V_j(q_j) \leqslant V_i\left(\sum_J q_j\right) \qquad \forall J \subset N \;\text{ and }\; \forall i \in J \tag{19}$$

which expresses that for risk sharing it is more efficient that market makers in J trade their equilibrium quantities, than if one of them undercut all the others.

Hence in equilibrium some risk sharing takes place, but only to a limited extent, as explained in the next subsection. Condition (18) also implies an upper bound on the transfers that the market makers can receive in equilibrium. If one market maker demanded too high a transfer, he would be undercut by one of his competitors, as soon as the revenue generated by undercutting would exceed the corresponding increase in risk bearing.

For the discussion of the next subsection, it is also useful to note that the competitive allocation $\{q_i^*\}_{\{i=1,\dots,N\}}$ and the competitive transfers $t_i^* = V_i'(q_i^*)q_i^*$ $\forall i \in N$ satisfy Condition (18). Actually, in this case, Condition (18) is equivalent to (using the fact that $V_i'(q_i^*) = V_j'(q_j^*)$ $\forall j \neq i$):

$$0 \leqslant V_i'(q_i^*)\left(\sum_{j\in J, j\neq i} q_j^*\right) \leqslant V_i\left(\sum_{j\in J} q_j^*\right) - V_i(q_i^*) \qquad \forall J \subset N \text{ and } \forall i \in J. \quad (20)$$

Since $V_i(.)$ is strictly convex, this condition is satisfied, hence the competitive outcome satisfies condition (18).

6.3. Equilibrium strategies

Condition (18) delineates the set of possible equilibrium outcomes but does not characterize the equilibrium strategies which can lead to those outcomes. We now compute explicitly such strategies. First we show that schedules involving simply two offers are sufficient to generate in equilibrium all the outcomes which satisfy (18). This implies that Condition (18) is not only necessary but also sufficient.

Proposition 10. *Consider an outcome* $\{(t_i, q_i)\}_{i=1,\dots,N}$ *which satisfies Condition* (18). *If the allocation rule is such that in case of a tie no dealer in the tie is excluded from the trade, then the following schedules based on two offers*

$$t_i(q_i) = t_i \quad \text{and} \quad t_i(D) = \sum t_j = F, \quad \forall i \in N$$

form an equilibrium.

Proposition 10 complements Proposition 8 and Proposition 9 by describing explicitly equilibrium strategies that give rise to the outcomes analyzed in these propositions. Remark that the equilibrium strategies described in Proposition 10 involve one 'serious offer', (t_i, q_i) which will be the equilibrium trade of the market maker and one out-of-the-equilibrium path threat $(\sum t_j, D)$ which ensures that the others will not raise their prices.[13] Note that, as exemplified in the

[13] A similar decomposition of equilibrium strategies can be found in Wilson (1979), Klemperer and Meyer (1989) and Back and Zender (1993).

numerical example presented in Section 3, these threats can give rise to quantity discounts in the equilibrium schedules posted by the market makers.

Thus, as in the two other market structures, Bertrand pricing is not guaranteed in equilibrium. In fact, as shown by Condition (18), there are many equilibria in which the market makers set prices above their reservation prices. Furthermore, in contrast with the limit order market, optimal risk sharing and competitive pricing are not granted as well. Actually, we have shown in the previous subsection that the competitive outcome satisfies Condition (18) but it is clear that this is not the only one. Moreover, note that the second inequality in Eq. (20) is not binding because $V_i(.)$ is strictly convex. This implies that even when the competitive allocation is obtained in equilibrium in the dealer market, the corresponding transfers can be higher than the competitive transfers. As will be shown in Section 6.5, when the allocation of risk is efficient transfers are likely to be very high. Finally, by showing that all outcomes satisfying condition (18) can be equilibrium outcomes, Proposition 10 shows that there is a large multiplicity of equilibria, which we analyze further in the next subsection.

This analysis shows that increasing the flexibility of the strategies available to the market makers (compared to the limit order market) can lead to a decrease in the efficiency of the market outcome. When market makers can use general schedules, they can threaten not to participate to the trade if they do not sell *a minimum quantity*. Thus they deter their competitors from improving upon their offers because if one market maker were to undercut the others, he would have to serve a large part of the whole demand. This is costly in term of risk bearing and quantity discounts make undercutting an even less attractive option (see the numerical example). Thus a combination of minima on the quantities they sell and quantity discounts can be used by the dealers to sustain high price equilibria, although they act non cooperatively. A similar phenomenon arises in the floor market. In this case, market makers are not encouraged to improve upon non competitive offers because the market maker who undercuts must serve the whole demand. Eq. (18) is reminiscent of Eq. (7), which characterizes the set of possible equilibrium prices in the case of the floor market.

On the contrary, in the limit order market, the market makers can neither propose quantity discounts (schedules are convex), nor impose minima on the quantity they sell (the buyer can always choose to buy from market maker i a quantity lower than $\bar{q}_i$). Consequently, this market structure is more conducive to undercutting. Hence the competitive outcome is obtained in the limit order market.

Note that in the dealer market equilibrium strategies are not limited to strategies involving simply two offers. For instance, there exists equilibria where schedules map each possible purchase, i.e., each real number in the interval $[0, D]$ into a transfer. For simplicity consider the case of two market makers. Denote $\{q_1^*, q_2^*\}$ the optimal allocation of risk between these two dealers. The following 'smooth' schedules form an equilibrium:[14]

$$\forall q \in [0, D],\ t_i(q) = V_i(q) + \Pi_i, \quad i = 1, 2$$

where $\Pi_1 = V_2(D) - V_1(q_1^*) - V_2(q_2^*)$, $\Pi_2 = V_1(D) - V_1(q_1^*) - V_2(q_2^*)$. Facing these schedules, the liquidity trader must trade q_1^* with market maker 1 and q_2^* with market maker 2. The total payment of the liquidity trader is then $F = V_1(D) + V_2(D) - V_1(q_1^*) - V_2(q_2^*)$. Consequently Conditions (a) and (b) in Proposition 8 are satisfied. Hence the schedules form an equilibrium. This equilibrium has the following properties:

- The first best allocation of risk is attained. Still there is a large spread (equal to Π_i, which is equal to the RHS of Condition (18)). The equilibrium ask price is larger than the reservation prices of the market makers, or the competitive price.
- The equilibrium schedules are discontinuous in zero, since $t_i(0^+) = \Pi_i > 0$. To purchase an infinitesimal but positive quantity from market maker i, the liquidity trader would have to pay the non-infinitesimal amount Π_i. This can be interpreted as a form of minima on the quantities sold by market makers, whereby they accept to trade small quantities only at very unattractive prices. This is in line with the empirical observation that, on SEAQ international, large trades obtain rather good prices, while small trades must incur a large spread. Note that this discontinuity in total transfers differs from the discontinuity in *marginal* transfers (the 'small trade spread') obtained in Glosten (1994). In the limit order market analyzed in Glosten (1994) infinitesimal trades correspond to infinitesimal transfers (and this is also the case in our analysis of the limit order market in Section 5).

6.4. Properties of the equilibria

We now discuss the properties of the equilibria arising in the dealer market. For simplicity we focus on the case of two market makers. In this case, from Propositions 9 and 10, equilibrium outcomes are characterized by

$$V_1(q_1) \leqslant t_1 \leqslant V_2(D) - V_2(q_2),$$
$$V_2(q_2) \leqslant t_2 \leqslant V_1(D) - V_1(q_1). \tag{21}$$

Fig. 2 illustrates these outcomes, which are discussed below:

- *Equilibrium multiplicity*: There is a large multiplicity of equilibria. This stems from coordination issues, reflecting strategic complementarity: the less

[14] This is similar to the 'truthful' strategies analyzed in Bernheim and Whinston (1986).

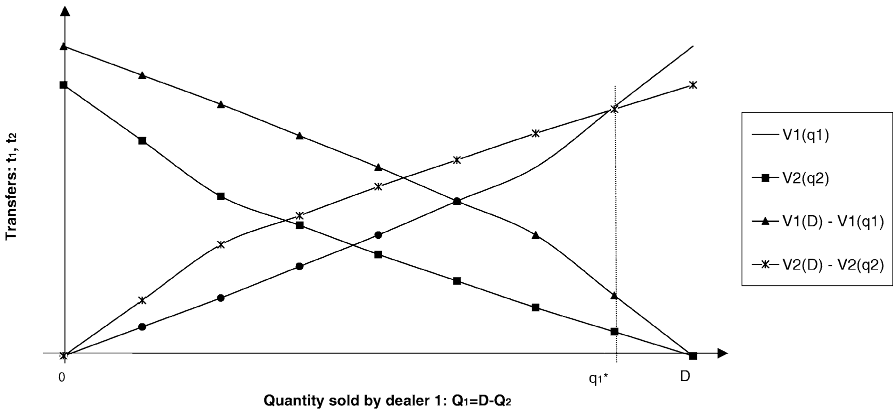

Fig. 2. Set of possible equilibrium outcomes with two dealers.

attractive the schedules offered by his competitors, the larger the residual demand market maker i faces, the larger the transfer he can demand for q_i. This generates equilibrium multiplicity.

- *Inefficient allocations*: As implied by Eq. (21), and as illustrated in the figure, all allocations $(q_i,\ q_j = D - q_i)$, $q_i \in [0, D]$ such that

$$V_i(q_i) + V_j(q_j) \leqslant \min(V_1(D),\ V_2(D)) \tag{22}$$

are possible equilibrium outcomes. As illustrated in the figure, because $V_1(D) > V_2(D)$, this precludes allocations whereby q_1 would be larger than q^* (defined graphically in the figure), i.e., whereby market maker 1 would serve nearly all the demand. Still the set of possible allocations is quite large, and contains many inefficient allocations, in addition to the unique efficient allocation. Again this reflects the presence of coordination issues: the market makers can coordinate on bad equilibria, i.e., inefficient allocations. This illustrates our earlier statement (in the discussion of Proposition 10) that the efficiency of risk sharing in this market structure is limited.

- *Non-competitive pricing*: Define unit prices as $p_i = t_i/q_i$. The unit prices at which the market makers sell the shares are such that

$$V_1(q_1)/q_1 \leqslant p_1 \leqslant \frac{V_2(D) - V_2(q_2)}{D - q_2},$$

$$V_2(q_2)/q_2 \leqslant p_2 \leqslant \frac{V_1(D) - V_1(q_1)}{D - q_1}. \tag{23}$$

Hence, for each possible allocation of trades, there are many possible prices, ranging from low reservation prices ($p_i = V_i(q_i)/q_i$), to large prices ($p_i = [V_j(D) - V_j(q_j)]/(D - q_j)$) at which the market makers earn large rents, and which are well above competitive prices.

- *Equilibrium price dispersion*: As can also be seen in Fig. 2 or from Eq. (23), in equilibrium the two market makers may well trade at different unit prices. Hence there can be equilibrium price dispersion although there is no asymmetric information, or differences in preferences among the buyers, or search costs. It is only due to the market structure and the strategic interactions between market makers it generates.

6.5. Selecting an equilibrium

In this section we investigate which of the many equilibria are more likely to be selected by the market makers.

6.5.1. Pareto dominant equilibrium

First we search for an equilibrium that would be Pareto dominant from the point of view of the market makers. In the case of two market makers (1 and 2), for given equilibrium trades q_1, q_2 there exists a Pareto Dominant equilibrium for the two market makers, the equilibrium which maximizes the transfers.[15] It yields transfers equal to

$$t_1 = V_2(D) - V_2(q_2),$$

$$t_2 = V_1(D) - V_1(q_1).$$

In this equilibrium expected utilities are

$$U_1(V_2(D) - [V_2(q_2) + V_1(q_1)]),$$

$$U_2(V_1(D) - [V_2(q_2) + V_1(q_1)]).$$

For these transfers, the equilibrium trade (q_1, q_2) which maximizes expected utility for both market makers is the efficient allocation, which maximizes risk sharing and leads to

$$V_1'(q_1) = V_2'(q_2).$$

[15] Note however that with more than two market makers there does not exist a Pareto dominant equilibrium, because of conflicts of interests between potential coalitions of market makers.

Note however that it is also the trade which maximizes transfers! We summarize these results in the next proposition.

Proposition 11. *Suppose there are two competing dealers. Among all the Nash equilibria of the game, the equilibrium, which from the point of view of both dealers is Pareto dominant, maximizes both the efficiency of trades and the price paid by the buyer.*

Hence the most efficient allocation is also the allocation which maximizes the payment for the buyer. This generates a conflict of interests between the market makers and the buyer. It also illustrates the trade-off between risk sharing efficiency and tightness of the spread, that we already had encountered in the floor market. The intuition is the following. By definition, in an efficient allocation, each market maker shares risk efficiently. Consequently, any deviation by one of the market makers, whereby he would undercut the others, is likely to lead to less efficient risk sharing, and therefore is not attractive for the market maker. Hence even if the market makers ask for large transfers they will not be undercut. Although we analyze non cooperative Nash equilibria, this association between risk sharing and high prices has a flavor of collusion.

6.5.2. Elimination of dominated strategies

Another way to identify equilibria which are particularly likely to emerge is to select equilibria by elimination of dominated strategies (EDS). Using this criterion leads us to consider offers that are not executed in equilibrium. Considering these offers is important in the present context, since they play an important role (as threats) in the construction of the equilibria.

For simplicity consider again the case of two market makers $i = 1, 2$. Consider an equilibrium where market maker i posts 2 take or leave it offers:

$$\{(t_i, q_i),(t_1 + t_2, D)\}.$$

Is this equilibrium robust to EDS? It is not if $t_1 + t_2 < V_i(D)$ – since in this case market maker i makes losses if he sells D, while he could have avoided those losses by not placing his order for D at $t_1 + t_2$. Thus robustness to EDS implies $F = t_1 + t_2 \geqslant \max(V_1(D), V_2(D))$. Hence the price paid by the buyer is larger than that paid in a second price indivisible quantity auction. This is striking, since risk bearing costs are lower in this market structure than in a second price indivisible good auction, and still the payment is higher.

6.6. Robustness of the results to the sequencing of the game

Our analysis is based on the assumption that the dealers make offers after they have been contacted by the liquidity trader, announcing his desired trade (D). Although this assumption is in line with the actual workings of dealer markets, one may wonder if our theoretical results would still obtain under the alternative assumption that the dealers move first, by quoting schedules, while the liquidity trader moves second, and selects how much to trade against these schedules. We definitely grant that it must be possible and interesting to construct richer models where other effects than those analyzed in our simple model arise. Yet, it is possible to offer a simple extension of the present model, whereby the order of the moves is reversed and similar results to those analyzed above obtain. This is illustrated in the following.

Suppose the liquidity trade can be $-2D$, $-D$, D, or $2D$, with probabilities $\Pr(-2D)$, $\Pr(-D)$, $\Pr(D)$, and $\Pr(2D)$ respectively. Assume for simplicity there are only 2 market makers and they have identical valuation functions: $V(q)$. For brevity focus on the ask side, the bid side is symmetric. As above suppose that, in case of ties, the liquidity trader favors allocations in which the two dealers participate rather than allocations excluding one of the dealers. In this context we obtain the following result.

Proposition 12. *In the simple extension of our model where the dealers post their schedules first and then the liquidity trader allocates her purchase, there exists a continuum of equilibria whereby each of the market makers posts a schedule offering to sell $D/2$ against transfer $t/2$, or D against transfer t, or $2D$ against transfer $2t$, where t is any real number such that*

$$V(D) < t < 2(V(D) - V(D/2)). \tag{24}$$

The continuum of equilibria described in Proposition 12 is qualitatively very similar to those arising in the case where the liquidity trader moves first. For example, Condition (21) is very similar to Condition (21). In particular Proposition 12 raises again the possibility that the dealers coordinate on a large spread equilibrium, with t close to $2(V(D) - V(D/2))$.

7. Conclusion

This paper analyzes how the differences in the types of offers liquidity suppliers can make lead to differences in trades, prices and risk-sharing. We consider three different quoting mechanisms. Large spreads and inefficient risk sharing arise in the market structure which offer the lowest flexibility (the floor market) and the largest flexibility (the dealer market). In the dealer market, we show that this uncompetitive outcome is due to bidding strategies involving minima on the quantity they sell and quantity discounts. As shown in empirical studies, such as Reiss and Werner (1995), these are indeed features of the pricing strategies used in dealer markets. In contrast, the competitive outcome is obtained in the limit order market, in which market makers can post convex schedules but are not allowed to impose minima on the quantity they sell and are not able to offer quantity discounts. This is consistent with the empirical findings in Christie and Schultz (1994a) that, in contrast with the findings obtained for a dealer market such as NASDAQ, there is no evidence of collusive pricing on limit order markets such as the NYSE or AMEX. This is also consistent with the finding by Huang and Stoll (1996) of larger spreads for a sample of NASDAQ stocks than for a matched sample of NYSE stocks.

Acknowledgements

Many thanks for helpful comments to seminar participants at the CentER, Carnegie Mellon University, Toulouse University, the Econometric Society Meetings, HEC, Pompeu Fabra University, the CEPR workshop on market microstructure in Barcelona, the NBER workshop on market microstructure, and the European Finance Association Meetings, especially Peter Bossaerts, Jean–Jacques Laffont, Lawrence Glosten, Christian Gollier, Oliver Hansch, David Martimort, Michel Moreaux, Jean Charles Rochet, Chester Spatt, Matt Spiegel (the editor), and Jean Tirole.

Appendix A. Proofs

Proof of Proposition 4. First suppose that market maker i and j are active, and $p_i < p_j$. Then i could raise slightly her price, and sell the same quantity $\bar{q}_i$ for a strictly higher profit, a contradiction. Thus at equilibrium all active market makers must quote the same price p. If market maker i (active or inactive) were to quote a price slightly under p, she could obtain a profit arbitrarily close to

$$\mathrm{Max}_{[0 \leqslant q \leqslant D]}\, pq - V_i(q)$$

by selling her competitive supply. Therefore inactive market makers must be such that their competitive supply at price p is zero; while active market makers must sell their competitive supply at price p. Since these offers sum to D, p must be the competitive price p^*.

The last point to check is whether an active market maker could profitably raise her price. The condition

$$\sum_{j \neq i, j \in \mathscr{A}} \bar{q}_j \geqslant D \quad \forall i \in \mathscr{A}$$

is sufficient to ensure that market maker i would have no demand in this case (non-indispensability); another possibility is that some inactive market makers quote a price higher but close to p, as to deter active market makers from raising their prices. □.

Proof of Corollary 5. We search for an equilibrium with the pro rata allocation such that no active market maker is indispensable because the condition $\sum_{j \neq i, j \in \mathscr{A}} \bar{q}_j \geqslant D$, $\forall i \in \mathscr{A}$ is satisfied. Then an equilibrium exists for the prorata allocation rule if for each competitive outcome: $\{q_i^*\}_{i \in N}$, there exists a set of limit quantities, $\{\bar{q}_i\}_{i \in N}$ such that if these limit quantities are offered then the prorata allocation rule yields the trades $\{q_i^*\}_{\{i \in N\}}$.

This is the case if:

$$0 = \begin{pmatrix} \left(\dfrac{q_1^*}{D} - 1\right) & \dfrac{q_1^*}{D} & \cdots & \dfrac{q_1^*}{D} \\ \dfrac{q_2^*}{D} & \left(\dfrac{q_2^*}{D}\right) - 1 & \cdots & \dfrac{q_2^*}{D} \\ .. & \cdots & \cdots & \cdots \\ \dfrac{q_N^*}{D} & \dfrac{q_N^*}{D} & \cdots & \left(\dfrac{q_N^*}{D}\right) - 1 \end{pmatrix} \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \\ .. \\ \bar{q}_N \end{pmatrix}$$

with the constraint that if $q_i^* = 0$ then $\bar{q}_i = 0$. This homogeneous system of linear equations admits an infinity of solutions. In the set of solutions, one must select the solutions such that the indispensability condition $\sum_{j \neq i, j \in \mathscr{A}} \bar{q}_j \geqslant D$, $\forall i \in \mathscr{A}$ is satisfied. □

Proof of Proposition 10. Consider an outcome $\{(t_i, q_i)\}_{\{i \in N\}}$, which satisfies Condition (18) and consider the strategies whereby the market makers announce two take or leave it offers,

$$t_i(q_i) = t_i \quad \text{and} \quad t_i(D) = F \quad \forall i \in N$$

Note that $q_i = 0$ and $t_i = 0$ if the outcome is such that market maker i is not active. We show that these strategies are an equilibrium because the pairs (t_i, q_i) satisfy Condition (18). First, note that the allocation $(q_1, \ldots, q_N)$ does indeed minimize the total payment for the buyer. The buyer might also choose to purchase all from one market maker only, but this is ruled out by the allocation rule.

Then in order to prove that these strategies form an equilibrium, we show that they satisfy Conditions (a) and (b) of Proposition 8. Condition (a) holds because the total amount F paid by the public is equal to $A_{-i}(D)$. Condition (b) requires that for all i, and all q:

$$A_{-i}(D - q_i) + V_i(q_i) \leqslant A_{-i}(D - q) + V_i(q). \tag{A.1}$$

It is shown in Section 6.1 that q on the RHS of Eq. (A.1) must be such that there exists a subset J of N (to which i belongs) such that $\sum_{j \in J} q_j = q$, so that the buyer can purchase exactly $D - q$. So, to establish that Eq. (A.1) holds, we only need to prove that, for all $J \subset N$ and $i \in J$

$$A_{-i}(D - q_i) + V_i(q_i) \leqslant A_{-i}\left(D - \sum_{j \in J} q_j\right) + V_i\left(\sum_{j \in J} q_j\right)$$

Now, this is equivalent to

$$\sum_{j \neq i} t_j + V_i(q_i) \leqslant \sum_{j \notin J} t_j + V_i\left(\sum_{j \in J} q_j\right)$$

$$\Leftrightarrow \sum_{j \neq i, j \in J} t_j + V_i(q_i) \leqslant V_i\left(\sum_{j \in J} q_j\right)$$

$$\Leftrightarrow \sum_{j \neq i, j \in J} [t_j - V_j(q_j)] \leqslant V_i\left(\sum_{j \in J} q_j\right) - \sum_{j \in J} V_j(q_j).$$

This is Condition (18). □

Proof of Proposition 12. Note that inequality (24) defines a non-empty interval of possible values for t, since, by convexity of V, $V(D)/2 > V(D/2)$.

First consider the case where the liquidity trader wants to buy D. In this case the trader splits his order and each dealer obtains utility: $t/2 - V(D/2)$. Should one of the market maker design his schedule to sell D in this case, by asking for this quantity a transfer just below t? If he did so he would get $t - V(D)$. Hence the condition under which there is no undercutting is $t/2 < V(D) - V(D/2)$. Finally note that the rationality condition of the dealers implies $t/2 > V(D/2)$. Hence the first equilibrium condition is

$$2V(D/2) < t < 2(V(D) - V(D/2)).$$

Of course this is exactly identical to the analysis of the previous subsections.

Second consider the case where the liquidity trader wants to buy $2D$. The condition under which no market maker undercuts his competitor to sell $2D$ is

$$t < V(2D) - V(D),$$

while the condition under which there is no undercutting to sell 1.5 D is

$$t/2 < V(1.5D) - V(D),$$

and the participation constraint is $t > V(D)$.

Combining these conditions and relying on the convexity of V, the result is established. $\square$.

References

Back, K., Zender, J., 1993. Auctions of divisible goods: on the rationale of the Treasury experiment. Review of Financial Studies 6, 733–764.

Bernheim, B.D., Whinston, M.D., 1986. Menu auctions, resource allocation and economic influence. Quarterly Journal of Economics 101, 1–31.

Biais, B., 1993. Price formation and equilibrium liquidity in fragmented and centralized markets. Journal of Finance. 48.

Biais, B., Hillion, P., Spatt, C., 1995. An empirical analysis of the limit order book and order flow in the Paris Bourse. Journal of Finance, 50.

Biais, B., Martimort, D., Rochet, J.C., 1997. Competing mechanisms in a common value environment, mimeo, Toulouse University.

Christie and Schultz, 1994a. Why do NASDAQ market makers avoid odd eighth quotes? Journal of Finance 49, 1813–1840.

Christie and Schultz, 1994b. Why did NASDAQ market makers stop avoiding odd eighth quotes? Journal of Finance 49, 1841–1860.

Dastidar, K.G., 1994. On the existence of pure-strategy Bertrand equilibrium. Economic Theory.

Dubey, P., 1982. Price-quantity strategic market games. Econometrica 50, 111–126.

Dutta, P., Madhavan, A., 1997. Competition and collusion in dealer markets. Journal of Finance 52, 245–276.

Glosten, L., 1994. Equilibrium in an electronic open limit order book. Journal of Finance 49, 1127–1161.

Grossman, S., 1981. Nash Equilibrium and the Industrial Organization of markets with large fixed costs. Econometrica 49, 1149–1172.

Hansch, O., Naik, N., Vishwanathan, S., 1997. Trading profits, inventory control and market share in a competitive dealership market. mimeo, London Business School.

Hasbrouck, J., Sofianos, G., 1993. The trades of market makers; an empirical analysis of NYSE Specialists. Journal of Finance 48, 1565–1594.

Ho, T., Stoll, H., 1983. The dynamics of dealer markets under competition. Journal of Finance 38, 1053–1074.

Huang, R., Stoll, H., 1996. Dealer versus auction markets: a paired comparison of execution costs on NASDAQ and the NYSE. Journal of Financial Economics, 313–357.

Klemperer, P.D., Meyer, M.A., 1989. Supply function equilibria in oligopoly under uncertainty. Econometrica 57, 1243–1277.

Lyons, R., 1995. Tests of microstructural hypotheses in the foreign exchange market. Journal of Financial Economics 39, 321–351.

Madhavan, A., Smidt, S., 1993. An analysis of daily changes in specialist inventories and quotations. Journal of Finance 48, 1595–1628.

Manaster S., Mann, S.C., 1996. Life in the pits: competitive marketmaking and inventory control, Review of Financial Studies 9, 953–976.

Reiss, P., Werner, I., 1995. Transactions costs in dealer markets: evidence from the London Stock Exchange. In: Lo, A. (Ed.), The Industrial Organization and Regulation of the Securities Industry, University of Chicago Press, pp. 125–176.

Simmons, Gerber, 1994. The bid-ask spread with shared trades. Working paper, York University.

Subrahmanyam, A., 1991. Risk-aversion, market liquidity and price efficiency. Review of Financial Studies 3, 417–441.

Wilson, R., 1979. Auctions of shares. Quarterly Journal of Economics 93, 675–689.